

# The point-splitting regularization of $(2+1)$ d parity breaking models

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The coefficient of the Chern-Simons term in the effective action for massive Dirac fermions in three dimensions is computed by using the point-splitting regularization method. We show that in this framework no ambiguities arise. This is related to the fact that the point-splitting regularization does not introduce additional parity breaking effects, implementing one possible physical criterion in order to uniquely characterize the system.

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## I. INTRODUCTION

It has long been known that three-dimensional gauge theories with parity or time reversal symmetry breaking display a set of far-reaching and interesting properties as, for instance, the celebrated mechanism for generating massive excitations without gauge symmetry breaking. Also, the behavior of  $(2+1)$  dimensional Dirac fermions in an electromagnetic background was extensively studied in the context of  $QED_3$  as a prototype to understand  $QED_4$  at finite temperature [1].

In particular, it is well established that the dynamics of  $(2+1)$ d massive Dirac fields induces a topological Chern-Simons term whose coefficient is not renormalized by higher order Feynman diagrams [2,3]. It is worth to point out that the one-loop effective action  $S_{eff}(s)$  corresponding to the three-dimensional fermionic determinant has been worked out by using several regularization methods [1,4–13]. The result however shows that a certain degree of ambiguity is present, due to the regularization dependence of the induced Chern-Simons coefficient  $\sigma$ , defined by the expressions

$$\begin{aligned} S_{eff}[s] &= \sigma S_{CS}[s] + \text{higher order terms}, \\ S_{CS}[s] &= \frac{1}{2} \int d^3x s_\mu \epsilon^{\mu\nu\rho} \partial_\nu s_\rho, \end{aligned} \quad (1.1)$$

where  $s_\mu$  is an external gauge field. Nonetheless, this coefficient plays a central role in the bosonization of  $(2+1)$ d fermionic systems and in the related applications to condensed matter, so that one is faced with the problem of its physical determination. We remind that the bosonization of the  $(2+1)$ d fermionic action  $K_F[\psi]$  for *free* massive fermions

$$K_F[\psi] = \int d^3x \bar{\psi} (i\rlap{\not{\partial}} + m) \psi, \quad (1.2)$$

is implemented in terms of a bosonizing gauge field  $A_\mu$ , while the current correlation functions are reproduced by the exact bosonization rule  $j^\mu = \bar{\psi} \gamma^\mu \psi \leftrightarrow \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$  [14–22]. Although a closed form for the bosonized action  $K_B[A]$  is not available, it turns out that  $K_B[A]$  is a gauge invariant functional whose leading term is the Chern-Simons action with coefficient given by  $\sigma^{-1}$ . This is the dominant term when a large mass expansion is considered.

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Recently, we have been able to show [21] that the above mentioned current bosonization rule is not only exact but is also universal. This means that when bosonizing a fermionic system containing current interactions  $I[j^\mu]$ , the correlation functions can be obtained from the mapping

$$K_F[\psi] + I[j^\mu] + \int d^3x s_\mu j^\mu \leftrightarrow K_B[A] + I[\epsilon^{\mu\nu\rho} \partial_\nu A_\rho] + \int d^3x s_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho. \quad (1.3)$$

This relationship has to be understood as an equivalence among the partition functions defined by the left (fermionic) and right (bosonic) sides, where  $K_B[A]$  is the functional corresponding to the bosonization of the *free* fermionic action  $K_F[\psi]$ .

In ref. [23], using the mapping (1.3), we studied transport properties in two-dimensional systems presenting a charge gap and displaying parity breaking properties. There, we related the universal rules (1.3) to the universal character of the transverse conductance  $I_t$  between two “perfect Hall regions”, that is, regions where the parity breaking parameter goes to infinity. We remind that in the relativistic case, the parity breaking parameter is the fermion mass itself. These perfect Hall regions were supposed to be adiabatically connected to regions containing arbitrary current interactions  $I[j^\mu]$ . In particular, we have shown that the transverse current between two “perfect Hall regions” is given in terms of the electric potential difference  $\Delta V$  between them, according to

$$I_t = \sigma \Delta V, \quad (1.4)$$

where  $\sigma$  is the induced Chern-Simons coefficient coming from the fermionic determinant. Notice that this result does not depend neither on the particular geometry of these regions, nor on the current interactions localized outside them. These considerations apply also to the nonrelativistic case, where the parity breaking parameter is given by the external magnetic field [23]. For example, when the first Landau band is completely filled, the induced Chern-Simons coefficient is unambiguously given by  $\sigma = 1/(2\pi)$ , which implies that the value, in usual units, of the universal Hall conductance is  $e^2/h$ . We underline that the topological information encoded in the mapping (1.3) [21], together with the particular parity breaking properties of the system, are all we need to derive the universal behavior of the transverse conductance. To some extent, the bosonization technique enhances the topological properties of the fermionic ground state.

Coming back to the relativistic case, as the induced Chern-Simons coefficient is related to a physical observable (the universal Hall conductance), further criteria have to be imposed in order to determine the *a priori* ambiguous value this coefficient can display. Indeed, from a purely theoretical point of view there is no way to decide which is the result, as there is no compelling theoretical reason to disregard a given regularization scheme on the basis of some serious inconsistency.

Therefore, the induced Chern-Simons coefficient has to be determined by additional physical requirements, to be specified according to the framework in which the relativistic fermionic action is being considered.

The action (1.2) is in fact extensively used in the context of the effective models describing the so called quantum critical transitions [24] and for the nodal quasiparticles [25].

In the first example, a fermionic field with Thirring-like interactions has been considered as a quantum critical model describing the topological transitions between plateaus in the Integer Quantum Hall Effect (IQHE) [24]. These transitions are characterized by a transverse conductivity  $\sigma_{xy} = (1/2n)(e^2/h)$  ( $n$  integer), and a longitudinal conductivity  $\sigma_{xx}$  which is *finite*, due to quantum fluctuations on all length scales. Indeed, any fermionic model displaying this behavior is called quantum critical; the value of the transverse conductivity defines the phase transition point.

The second example corresponds to nodal liquid models for high  $T_c$  superconductors [25]. The excitations here are described by a couple of  $(2+1)$ d Dirac fermions. Depending on the order parameter used to describe the  $d$ -wave superconductor ( $d_{x^2-y^2}$  or  $d_{x^2-y^2} + id_{xy}$ ), the system does (or does not) break parity and time reversal symmetry.

In both examples, a central property characterizing the physical system is the amount of parity symmetry breaking, which plays a fundamental role in order to construct the corresponding effective lagrangians and which should be considered as establishing a possible set up for the determination of the model. In other words, we should adopt here the criterion of not introducing additional parity breaking, whenever the theoretical possibility of ambiguous results shows up at the quantum level. In fact, if this is to be a possible physical criterion for the determination of the system, then, any regularization scheme having no additional sources of parity breaking should lead to the same result.

The aim of this paper is twofold. First, we address the issue of the determination of the coefficient  $\sigma$  within the framework of the point-splitting regularization, which belongs to the above mentioned class of regularization schemes. We will follow the method presented in ref. [26], where the point-splitting was successfully applied to study anomalies in non-abelian chiral gauge theories. Among the schemes which do not introduce additional parity breaking, the point-splitting turns out to be particularly adapted to the present case, as it combines at any stage many desirable properties. It can be implemented at the lagrangian level, and preserves translation and Lorentz symmetry, as well as

invariance under *small* gauge transformations. We will see that the obtained Hall conductance equals half the perfect value ( $\frac{1}{2} \frac{m}{|m|} e^2/h$ ), namely  $\sigma = \frac{1}{4\pi} \frac{m}{|m|}$ , as it is generally assumed in the framework of condensed matter systems [27].

Second, we have tried to collect a large amount of information about the coefficient  $\sigma$ , comparing the results obtained by different regularizations. This point should be of some usefulness in order to have a general view of the situation concerning  $\sigma$ , helping in clarifying the possible physical criterion to be adopted for its determination. Although in the following we shall restrict ourselves to the zero temperature case, we shall also mention briefly the important progress which has been recently achieved for nonvanishing temperature [28–30].

The paper is organized as follows. In section §II we report on different regularization methods. In section §III we analyze the relationship between the coefficient  $\sigma$  and parity properties. Section §IV is devoted to the evaluation of the induced Chern-Simons coefficient by using the point-splitting. Finally, in section §V, we present the conclusions.

## II. REGULARIZATION AMBIGUITIES

The simplest example for a  $(2+1)$ d model with parity breaking properties is the massive Dirac fermion model. In this case, the parity breaking parameter is the fermion mass; indeed, under a parity transformation  $\mathcal{P}$ , the lagrangian density transforms according to

$$\bar{\psi} (i\not{D} + m) \psi \xrightarrow{\mathcal{P}} \bar{\psi} (i\not{D} - m) \psi. \quad (2.1)$$

As a consequence, the fermionic effective action, which gives the system's response to the external electromagnetic field  $A_\mu$ , is expected to contain a parity breaking term, that is, an induced Chern-Simons term. Moreover, the induced Chern-Simons coefficient should be naively expected to be related to the sign of the fermion mass, since the parity breaking is not related with the absolute value of  $m$  but with its sign (see eq. (2.1)). However, the presence of superficially linear ultraviolet divergences, when computing the one-loop effective action, require the introduction of some regularization scheme. Upon a closer look, the induced Chern-Simons coefficient turns out to be finite but ambiguous, depending on the particular way we regulate the divergences. Any induced Chern-Simons term in the effective action whose coefficient is not related with the sign of the mass is called an *anomalous term*, in the sense that it represents an additional parity breaking which is not initially present in the classical fermionic action.

Another symmetry, that could be expected to be present in the fermionic effective action, is the invariance under large gauge transformations, which arise when the euclidean time coordinate is compactified to a circle in order to deal with finite temperature. This symmetry would follow from the path integral definition of the effective action, where all fermion field configurations are considered, including those corresponding to large gauge transformations. In this context, a series of recent articles have shown that anomalous terms, together with non-extensive parity breaking terms, are required in order to preserve large gauge invariance of the finite temperature effective action [28–30].

Let us proceed thus by reviewing the results obtained by different regularization schemes.

### Regularization schemes

#### a) Dimensional regularization

The dimensional regularization has been employed by the authors of ref. [6]. The result is given by

$$\sigma = \frac{1}{4\pi} \frac{m}{|m|} \quad (2.2)$$

We observe that the same result has been obtained by the differential regularization [11].

#### b) $\zeta$ -function regularization

The  $\zeta$ -function regularization is based on the calculation of the fermionic current, by means of a regularized Dirac Green function. To this aim, the regularized current is written as

$$J_\mu^{\text{reg}} = -\frac{d}{d\lambda} \left\{ \lambda \text{Tr} \left[ \gamma_\mu (\not{\partial} + \not{\epsilon} + m)^{-\lambda-1} \right] \right\} \Big|_{\lambda=0} \quad (2.3)$$

where  $\lambda$  is a complex variable to be analytically continued to  $\lambda = 0$ , where the Green function is not well defined. Due to subtleties in the analytic continuation, the result contains extra parity breaking anomalous terms which are not associated with the sign of the original fermion mass. In this case the result is [7,8]

$$\sigma = \frac{1}{4\pi} \left( \frac{m}{|m|} \pm 1 \right) \quad (2.4)$$

These two possibilities have been related to a determination of the system compatible with invariance under large gauge transformations, when the euclidean time coordinate is compactified to live on a circle  $S^1$ .

c) *Pauli-Villars regularization*

The induced Chern-Simons coefficient evaluated by means of the Pauli-Villars regularization was presented in several places [5]. The result obtained is

$$\sigma = \frac{1}{4\pi} \left( \frac{m}{|m|} + q \right), \quad (2.5)$$

where  $q$  is an integer. It turns out that in the abelian theory it is possible to choose  $q = 0$ , to all orders in a perturbative expansion, provided we take an appropriate coupling constant renormalization [31].

It is worth spending here some words about the regularization ambiguity present in the expression for  $\sigma$ . In fact, the Pauli-Villars regularization is expected to be related with a higher order derivative regularization, where each regulating mass parameter is associated to a new Dirac factor that changes the fermion propagator, so as to regularize the ultraviolet behavior, namely

$$(i\cancel{D} + m)^{-1} \rightarrow \left[ (i\cancel{D} + m) \prod_i (i\cancel{D} + \Lambda_i)/\Lambda_i \right]^{-1}. \quad (2.6)$$

In general, this procedure changes the parity properties of the starting action. Each regulating mass  $\Lambda_i$  should contribute an additional term  $q_i = \Lambda_i/|\Lambda_i|$  to the induced coefficient. The change in the parity properties of the initial theory is measured by  $q = \sum_i q_i$  (cf. eq.(2.5)). Then, if we are interested in working with an effective action having the same parity properties of the classical fermionic field theory, we should consider an equal number of positive and negative regulating masses. This would correspond to set  $q = 0$  in eq.(2.5). Note also that the Dirac factor associated to a pair of regulating masses  $\Lambda$  and  $-\Lambda$ ,

$$(i\cancel{D} + \Lambda)(i\cancel{D} - \Lambda), \quad (2.7)$$

(which corresponds to  $q = 0$ ) is invariant under a parity transformation, *i.e.*, it does not introduce additional parity breaking. In this case, if the external field coupling is maintained to be  $A_\mu \bar{\psi}(x) \gamma^\mu \psi(x)$ , a systematic iterative procedure to recover gauge invariance order by order in perturbation theory must be considered [13].

d) *Lattice regularization*

When the ultraviolet divergences are regulated by defining the theory on a lattice, the induced coefficient is [12]

$$\sigma = \frac{1}{4\pi} \frac{m}{|m|} n \quad (2.8)$$

where  $n$  is an arbitrary integer, identified with a topological number [12]. It turns out that  $n$  is the winding number that appears when the fermion propagator (in momentum representation) is viewed as a mapping from a 3-dimensional torus onto the space of  $SU(2)$  matrices.

Notice that this result coincides with the Pauli-Villars regularization, although the interpretation for the integer  $n$  is very different. While in the later case it is easier to relate the ambiguous coefficient with additional parity breaking, in the former one, the ambiguity is related to the different possible inequivalent formulations for fermions on the lattice.

### III. PHYSICAL DETERMINATION OF THE SYSTEM

In the introduction, we have stressed that the universal transverse conductance is identified with the induced Chern-Simons coefficient  $\sigma$  in the fermionic effective action (cf. eq.1.4). However, as discussed in the previous section, this coefficient is affected by ambiguities which have to be fixed by a suitable physical criterion. To this aim, we discuss in

this section the relationship between  $\sigma$  and parity. Before going any further, let us underline a key property concerning the ambiguity, namely, all the results have the following form

$$\sigma = \frac{1}{4\pi} \times \text{integer} \quad (3.1)$$

whatever the particular regularization scheme is, suggesting thus a topological origin for  $\sigma$ . In fact, as shown in [9], the Ward identity for small gauge invariance allows to relate  $\sigma$  to a topological invariant which has the form of a Wess-Zumino term, implying the eq.(3.1). This is a highly nontrivial result for a physical quantity. We are dealing therefore with an ambiguity which is of an unusual type as compared to the ordinary field theory ambiguities associated to the genuine ultraviolet divergences, fixed by a set of renormalization conditions.

For a better understanding of the relationship between  $\sigma$  and parity, we recall here that condensed matter effective models containing  $(2+1)$ d fermions are usually defined by implementing, at the lagrangian level, the parity breaking properties of the system. In particular, information about transport properties can be obtained from the associated equations of motion. For instance, it is well known that the field modes of the Dirac equation

$$(i\mathcal{D} + m)\psi = 0, \quad (3.2)$$

can be quantized to compute a transverse conductance or, equivalently, a proportionality factor  $\sigma$  between charge and flux [32,33]. This follows from the fact that the spectrum of the eq.(3.2) in the presence of an external magnetic flux  $\Phi$  displays an asymmetry related to the presence of zero modes, whose degeneracy is  $\Phi/(2\pi)$  [34]. As a consequence, the vacuum expectation value of the charge operator  $Q$  receives contributions only from the zero modes, and is given by [32,33]

$$\langle Q \rangle = \frac{1}{2} \frac{m}{|m|} \Phi/(2\pi) \quad , \quad Q = \frac{1}{2} \int d^2x [\psi^\dagger, \psi]. \quad (3.3)$$

This corresponds to a proportionality factor between charge density and magnetic field (or transverse conductance)  $\sigma = \frac{1}{4\pi} \frac{m}{|m|}$ . Note that the only parity breaking effects are those already present in the Dirac lagrangian (cf. eq.(3.2)).

This result may be interpreted as enforcing the criterion of defining the model at quantum level by not introducing additional parity breaking effects. For this criterion be well defined, any compatible regularization scheme should lead to the same transverse conductance. Observe indeed that the Pauli-Villars or the higher order regularization with  $q = 0$  agree with the zero mode calculation.

To further exploit this idea, in the next section we will compute the induced Chern-Simons coefficient by following the point-splitting method, which enjoys the property of not introducing additional parity breaking. Therefore it will provide a nontrivial check for the determination of  $\sigma$ .

#### IV. THE INDUCED CHERN-SIMONS COEFFICIENT AND THE POINT-SPLITTING REGULARIZATION

Let us start by considering the lagrangian density, defined on a  $(2+1)$ d space-time,

$$\mathcal{L} = i\bar{\Psi}(x)\gamma^\mu\partial_\mu\Psi(x) + \bar{\Psi}(x)\gamma^\mu A_\mu\Psi(x) - m\bar{\Psi}(x)\Psi(x), \quad (4.1)$$

where  $A_\mu$  is an external gauge field, and the matrices  $\gamma_\mu$  are defined in terms of the Pauli matrices,  $\gamma^0 = \sigma^3$ ,  $\gamma^1 = i\sigma^1$ ,  $\gamma^2 = i\sigma^2$ . With this definition  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$  where  $g_{\mu\nu} = \text{diag}(1, -1, -1)$ .

The essence of the point-splitting regularization is to split the product of local operators by means of the introduction of a small vector  $\epsilon_\mu$ . It is clear that this procedure modifies the ultraviolet (short distance) behavior of the theory. When choosing a particular space-time direction, Lorentz invariance is broken. However, this symmetry can be recovered by properly averaging the products over all possible  $\epsilon_\mu$  orientations.

The regularized lagrangian density is defined by [26]

$$\mathcal{L}_{\text{reg}} = -2i\overline{\left\{\bar{\Psi}(x+\epsilon)\frac{\gamma\cdot\epsilon}{\epsilon^2}T\left(e^{i\int_{-1}^1 dt\epsilon_\mu A^\mu(x+\epsilon t)}\right)\Psi(x-\epsilon)\right\}} - m\bar{\Psi}(x)\Psi(x), \quad (4.2)$$

where  $T$  denotes a time-ordered product with respect to the variable  $t$ . The bar over the first term in eq.(4.2) represents the average over the orientations of the vector  $\epsilon_\mu$ . It is easy to show that

$$\lim_{\epsilon \rightarrow 0} \mathcal{L}_{\text{reg}} = \mathcal{L}. \quad (4.3)$$

We also see that expression (4.2) preserves gauge and Lorentz invariance, while a parity transformation only acts by changing the sign of the fermion mass. In fact, if a parity transformation in the first term of eq. (4.2) is considered, one of the components of  $\epsilon_\mu$  will change sign, say  $\epsilon_1 \rightarrow -\epsilon_1$ . However, this change has no effect because of the  $\epsilon_\mu$  averaging process. Thus, the regularized lagrangian  $\mathcal{L}_{\text{reg}}$  shares the same symmetry properties of the original unregularized lagrangian.

Following [26], it is convenient to rewrite the action  $A = \int d^3x \mathcal{L}_{\text{reg}}(x)$  in momentum space

$$A = \int \frac{d^3p}{(2\pi)^3} \bar{\Psi}(p) S^{-1}(p) \Psi(p) + \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \bar{\Psi}(p + \frac{1}{2}q) \Gamma(p, q) \Psi(p - \frac{1}{2}q), \quad (4.4)$$

where

$$S^{-1}(p) = -2i \frac{\overline{\gamma \cdot \epsilon}}{\epsilon^2} e^{2ip \cdot \epsilon} - m = -\gamma_\mu \frac{\partial}{\partial p_\mu} \left( \frac{e^{2ip \cdot \epsilon}}{\epsilon^2} \right) - m. \quad (4.5)$$

The interactions are encoded in the function

$$\Gamma(p, q) = \sum_{n=1}^{\infty} \Gamma^{(n)}(p, q) \quad (4.6)$$

with  $\Gamma^{(n)}$  containing the  $n^{\text{th}}$  power of the gauge field

$$\Gamma^{(n)} = \frac{1}{(2\pi)^{3n} n!} \int \left( \prod_{j=1}^n dk_j \right) \delta(q - \sum_{l=1}^n k_l) T \left\{ \int_{-1/2}^{1/2} \prod_{r=1}^n dt_r \left[ \prod_{s=1}^n A(k_s) \cdot \frac{\partial}{\partial p} \right] S^{-1}(p - \sum_{g=1}^n k_g t_g) \right\}. \quad (4.7)$$

We note that a particular average of  $[exp(2ip \cdot \epsilon)]/\epsilon^2$  in eq. (4.5) corresponds to fixing the form of  $S^{-1}(p)$ . In the case where the average enforces Lorentz invariance, we can use the following ansatz

$$\frac{\partial}{\partial p_\nu} \left( \frac{e^{2ip \cdot \epsilon}}{\epsilon^2} \right) = \frac{p^\nu}{f(\mu, p^2)}, \quad (4.8)$$

where  $\mu$  is a regularization parameter and  $f(\mu, p^2)$  is an arbitrary analytic function of  $p^2$  satisfying

$$\lim_{\mu \rightarrow 0} f(\mu, p^2) = 1. \quad (4.9)$$

In terms of  $f(\mu, p^2)$ , the regularized propagator reads

$$S(p) = f(\mu, p^2) \frac{\not{p} + m f(\mu, p^2)}{p^2 - m^2 f^2(\mu, p^2)}. \quad (4.10)$$

Notice that, in the limit where  $\mu \rightarrow 0$ , the free massive Dirac propagator is correctly reobtained. Also, using eqs.(4.7) and (4.10) we see that the asymptotic form of  $\Gamma^{(n)}$  is

$$\Gamma_{p \rightarrow \infty}^{(n)}(p, q) \approx \frac{1}{f(\mu, p^2) p^{n-1}}. \quad (4.11)$$

This equation, together with eq.(4.10), implies that all loop integrals can be made convergent by choosing a function  $f(\mu, p^2)$  with a fast enough growing behavior as  $p \rightarrow \infty$ . Moreover, from eq. (4.11), we can see that among the various  $\Gamma^{(n)}$  ( $n = 1, 2, \dots$ ),  $\Gamma^{(1)}$  has the lowest decreasing degree as  $p \rightarrow \infty$ . Therefore, if a particular Feynman diagram  $G$  containing  $\Gamma^{(1)}$  as a vertex corresponds to a finite loop integral, any diagram obtained from  $G$  by replacing  $\Gamma^{(1)} \rightarrow \Gamma^{(n)}$  ( $n = 2, 3, \dots$ ) will be associated with a finite loop integral. Thus, in order to determine the function  $f(\mu, p^2)$  only the diagrams containing the vertex  $\Gamma^{(1)}$  need to be considered. A simple power counting argument shows that if  $f(\mu, p) \approx p^4$  as  $p \rightarrow \infty$ , then all loop integrals turn out to be finite. These requirements can be fulfilled by choosing, for instance,  $f(\mu, p) = 1 - (\mu/m)^2 p^2 + (\mu/m)^4 p^4$ . However, we will proceed by considering a general form of  $f(\mu, p^2)$  compatible with the convergence conditions.

As is well known, the Chern-Simons coefficient is completely determined by the one loop contribution, and is related to the vacuum polarization tensor  $\Pi_{\mu\nu}$ . Let us consider then the vacuum functional

$$e^{iW(A)} = \langle T[\exp(iA_I)] \rangle_0, \quad (4.12)$$

where  $A_I$  is the interaction part of the action. In order to compute the vacuum polarization  $\Pi_{\mu\nu}$ , we look for the quadratic contribution in the gauge fields to the effective action  $W(A_\mu)$ , given by

$$W^{(2)} = \langle T[(iA_I)^2/2] \rangle_0. \quad (4.13)$$

Here, in the expression of  $A_I$ , all we need to consider is the vertex  $\Gamma^{(1)}$ . The linearized interaction reads

$$A_I = -2i \int dx \left\{ \overline{\Psi}(x + \epsilon) \frac{\gamma \cdot \epsilon}{\epsilon^2} \times \left[ i \int_{-1}^1 dt \epsilon \cdot A(x + \epsilon t) \right] \Psi(x - \epsilon) \right\}. \quad (4.14)$$

Using this equation and Wick's theorem, we can rewrite  $W^{(2)}$  in the form

$$W^{(2)} = \frac{1}{2} \int dx_1 dx_2 \text{Tr} \left\{ \overline{\left( (-2i \frac{\gamma \cdot \epsilon}{\epsilon^2}) \left[ \int_{-1}^1 dt_1 \epsilon \cdot A(x_1 + \epsilon t_1) \right] \langle \overline{\Psi}(x_1 + \epsilon) \Psi(x_2 - \epsilon) \rangle_0 \right)} \times \right. \\ \left. \overline{\left( (-2i \frac{\gamma \cdot \epsilon}{\epsilon^2}) \left[ \int_{-1}^1 dt_2 \epsilon \cdot A(x_2 + \epsilon t_2) \right] \langle \overline{\Psi}(x_2 + \epsilon) \Psi(x_1 - \epsilon) \rangle_0 \right)} \right\}, \quad (4.15)$$

where  $\langle \overline{\Psi}(x_1 + \epsilon) \Psi(x_2 - \epsilon) \rangle_0$  and  $\langle \overline{\Psi}(x_2 + \epsilon) \Psi(x_1 - \epsilon) \rangle_0$  denote the free fermion propagators written in coordinate space.

An explicit evaluation of the mean value leads to the following momentum space result

$$W^{(2)} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} A^\mu(q) \Pi_{\mu\nu}(q) A^\nu(-q), \quad (4.16)$$

where the regularized vacuum polarization tensor reads

$$\Pi_{\mu\nu}(q) = -\text{Tr} \left\{ \int \frac{d^3 p}{(2\pi)^3} \int_{-1/2}^{1/2} dt_1 dt_2 \left[ \frac{\partial}{\partial p^\mu} \left( \frac{\not{p}}{f(\mu, u^2)} \right) \frac{\not{p} + m f(\mu, p^2)}{p^2 - m^2 f^2(\mu, p^2)} f(\mu, p^2) \right] \times \right. \\ \left. \left[ \frac{\partial}{\partial p^\nu} \left( \frac{\not{p}}{f(\mu, v^2)} \right) \frac{\not{p} + \not{q} + m f(\mu, (p - q)^2)}{(p - q)^2 - m^2 f^2(\mu, (p - q)^2)} f(\mu, (p - q)^2) \right] \right\} \quad (4.17)$$

and  $u, v$  are the vectors

$$u_\nu \equiv p_\nu - q_\nu t_1 \quad (4.18)$$

$$v_\nu \equiv p_\nu - q_\nu + q_\nu t_2. \quad (4.19)$$

The expression (4.17) can be split into a symmetric and an antisymmetric part. The symmetric part follows from the parity conserving terms in the effective action and turns out to be regularization independent; a closed expression can be found in [1]. On the other hand, the antisymmetric part is related to the parity breaking terms. In general, it depends on the regularization through the induced local Chern-Simons coefficient

$$\sigma_{(\mu)} = \frac{1}{2} \lim_{q^2 \rightarrow 0} \epsilon_{\mu\nu\rho} \frac{q^\rho}{q^2} \Pi^{\mu\nu}(q^2). \quad (4.20)$$

Performing the trace over the spin degrees of freedom, evaluating the integrals over  $t$ , and taking the limit for small momenta, the regularized induced Chern-Simons coefficient is found to be

$$\sigma_{(\mu)} = mF(m, \mu), \quad (4.21)$$

where

$$F(m, \mu) = \frac{2}{3} \int \frac{d^3 p}{(2\pi)^3} \frac{4\mu(p^2/m^2)(1 - 2\mu^3 p^2/m^2) + 3}{f(\mu, p^2)[p^2 - m^2 f^2(\mu, p^2)]^2}. \quad (4.22)$$

Observe that the integral in the eq. (4.22) is finite for all values of  $\mu$ . Therefore, using eq. (4.9), we can take the limit  $\mu \rightarrow 0$  in the integrand, and perform the momentum integral to obtain

$$\lim_{\mu \rightarrow 0} \sigma(\mu) = \sigma = \frac{1}{4\pi} \frac{m}{|m|}. \quad (4.23)$$

We remark that this result does not depend on the explicit form used for the function  $f(\mu, p^2)$ . The whole calculation relies on two properties, namely, at large momenta  $f(\mu, p^2)$  grows fast enough so as to regularize the theory and  $\lim_{\mu \rightarrow 0} f(\mu, p^2) = 1$ .

Another important point to be mentioned is that we have taken the limit  $\mu \rightarrow 0$  in the integrand of eq. (4.22). This is completely justified since, as observed before, the integral (4.22) is *finite*, for any value of  $\mu$ . This is not the case, for instance, of the chiral anomaly in  $(3+1)d$  where the loop integrals are divergent and it is not possible to exchange the order of the integration with the limit  $\mu \rightarrow 0$  [26].

## V. CONCLUSIONS

Following the point-splitting regularization method, we have computed the induced Chern-Simons coefficient. The latter is seen to be independent of the particular splitting averaging process, taking the unambiguous value  $\sigma = \frac{1}{4\pi} \frac{m}{|m|}$ .

Our main motivation for the preceding calculation was that of considering a well defined regularization scheme which does not break parity. In particular, the point-splitting can be implemented at the lagrangian level maintaining, at any stage, translation, Lorentz and *small* gauge symmetry.

This calculation supports the idea that a possible physical criterion for the determination of the fermionic system can be that of not introducing additional parity breaking effects. This physical determination is natural in most condensed matter models that include  $(2+1)d$  relativistic fermions, as those describing quantum critical transitions and nodal liquids.

We also note that the same result for  $\sigma$  is obtained by counting the zero modes of the Dirac equation in the presence of an external magnetic flux [33].

To some extent, this determination can be compared with similar behavior in  $(1+1)d$  systems. There, because of charge conservation, the natural choice is not breaking gauge symmetry in most models containing chiral anomalous  $(1+1)d$  fermions, as those describing Luttinger liquids in quantum wires. We note that the anomalous terms in the effective action for  $(2+1)d$  fermions can also be understood à la Fujikawa [35].

Finally, other physical determinations of the fermionic system cannot of course be disregarded. Among the alternative possibilities, the requirement of large gauge invariance, when the Euclidean time coordinate is compactified to live on a circle, is particularly interesting.

This determination could be relevant when discussing the construction of Green's functions for the vortex excitations present in the bosonized  $(2+1)d$  theories. These excitations could be created by the introduction of monopole singularities, which lead to a quantization of the Chern-Simons coefficient, compatible with large gauge invariance [36]. This context would be analogous to that presented in ref. [37], where skyrmion Green's functions are defined by the introduction of instanton singularities. In both cases, the time compactification can be associated to fixed boundary conditions around the singularities.

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